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Mixed-Integer Optimal Control of a Residential Heating Network using Linear and Nonlinear Programming Techniques

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***Abstract*— This work investigates the solution of Mixed-Integer Optimal Control Problems (MIOCPs) for an existing residential heating network. The network consists of several buildings that are interconnected through a district heating network and a biomethane combined heat and power (CHP) plant. All buildings have access to decentralized heat generation, in the form of solar thermal collectors on the rooftops of the buildings. Buildings with surplus heat are intended to transfer heat to buildings with heating demands in order to prevent the activation of the CHP plant. Additional storage provide further flexibility for storage and utilization of heat. Binary variables represent the exchange relations between buildings and the CHP plant. For this system, we solve an MIOCP in two different ways. On the one hand, we keep the system related nonlinearities and apply the Combinatorial Integral Approximation (CIA) method to the arising Mixed-Integer Nonlinear Program (MINLP). On the other hand, we apply a linear reformulation yielding a Mixed-Integer Linear Program (MILP), which we solve using a standard MILP solver. We show that the MINLP approach has a computational advantage over the MILP approach, while yielding only slightly worse results in the single-digit percentage range for selected key figures.**

I. Introduction

The building sector has a significant influence on the achievement of national and international climate targets. In Germany for example, emissions in this sector must be reduced by 40 % by 2030 in order to achieve climate targets [1]. Heating networks based on renewable energies can achieve significant savings in energy consumption and reductions in emissions compared to conventional heating networks [2], [3]. In addition to increasing the energy efficiency of buildings by, e.g. insulating walls and refurbishing windows, smart operation of the heating network and its components contribute to strengthening the benefits [4].

District heating networks often incorporate so-called switched systems. Such systems are characterized by having continuous-time dynamics and discrete switching events [5]. The operational optimization then leads to the formulation of Mixed-Integer Optimal Control Problems (MIOCPs). In order to solve MIOCPs we make use of direct methods as these are considered to be preferable to indirect methods [6]. For the case of MIOCPs this leads to Mixed-Integer Linear Programs (MILPs) or Mixed-Integer Nonlinear Programs (MINLPs), depending on the problem formulation.

A. Related Work

MINLPs are generally much more difficult to solve than MILPs. However, the system dynamics may cause a non-linear formulation, which then require to either solve a resulting MINLP directly, or to transform it into an MILP. A common nonlinearity for heating applications is, e.g. the bilinear relationship between a temperature T and a mass flow $\dot{m}$ in energy balances.

To obtain linear approximations of such relations within formulation of MILPs, e.g. the McCormick relaxation or other (convex) transformation methods can be applied [7], [8], [9]. Other approaches may avoid nonlinearities by application of piece-wise linear approximations [10]. What unites all these approaches is that the reformulation leads to a non-negligible increase in the number of optimization variables and constraints, which may offset the advantages of linear functions [10].

Furthermore, application of such techniques for linearization of the original MINLP may not always be applicable. In such case, it is typically not favorable in an optimal control setting to apply a general MINLP solver [6]. Instead, tailored solution methods have been presented [6], [11] that facilitate approximate solution of the original MINLP to obtain a fast, however, possibly suboptimal solution to the problem.

Within this work, we solve a MIOCP for optimization of the heat exchange in a residential heating network, once using a MINLP formulation and once using a MILP formulation of the problem. For obtaining a fast approximate solution of the MINLP, we apply the Combinatorial Integral Approximation (CIA) method [11]. The MILP formulation is solved using a standard MILP solver. We show that the MINLP approach has a computational advantage over the MILP approach, while yielding only slightly worse results regarding selected key figures.

B. Structure of the paper

The paper is organized as follows. The next section introduces the residential heating network model, where we state the system description and model setup. In Section III, the optimal control problem formulation applied to our model is explained in more detail, followed by descriptions of the applied methods. Afterwards the results are shown in Section IV and the paper is wrapped up with the conclusion.

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II. THE RESIDENTIAL HEATING NETWORK

A. System description and objective

The work is inspired by a real heating network located in Freiburg, Germany [12]. A sketch of the network considered within this paper is given in Fig. 1. The heating network consists of several residential buildings with distributed heat generation and storage facilities. Each building accommodates several housing units and is equipped with solar thermal collectors (STCs) for heat production on their rooftops. The decentralized heat production varies, depending on the size of the buildings and amount of housing units. The STC yield may be used directly for self-consumption, to charge the storage tank, or to be supplied to another household within the network. The flexibility of households further stems from the storage options. The total storage capacity in the network exceeds the capacity of the pipe network.

The residential heating network can be controlled by a central unit and operated in two modes. On the one hand, we speak of the household heating network exchange mode (HM), when the residential buildings transfer heat among themselves. On the other hand, if the biomethane combined heat and power (CHP) plant is in operation, we refer to it as the district heating network mode (DM). Both modes, HM and DM, use the same pipes for the heat transfer. Note that the buildings are not able to feed heat into the grid without another household receiving it (indicated by the arrows in Fig. 1). As a limitation, both network modes cannot be operated at the same time, i.e., if households collaboratively exchange heat we say that the household heating network is active and the heat distribution by the CHP plant is turned off, and vice versa. The heating network furthermore reacts slowly to changes, which is why there is a minimum requirement for the quantity, or temperature difference, in any heat exchange. What also prevents comparatively high heat losses. The proposed MIOCP formulation should favor the heat exchange between households and coverage via the storages over a request for heat from the CHP plant.

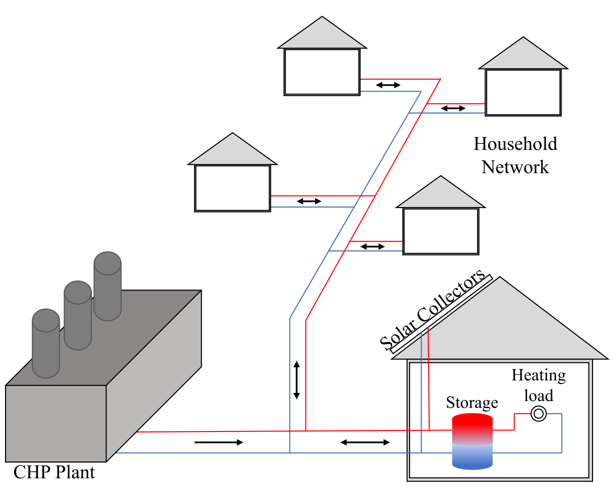

Fig. 1. Sketch of the residential heating network.

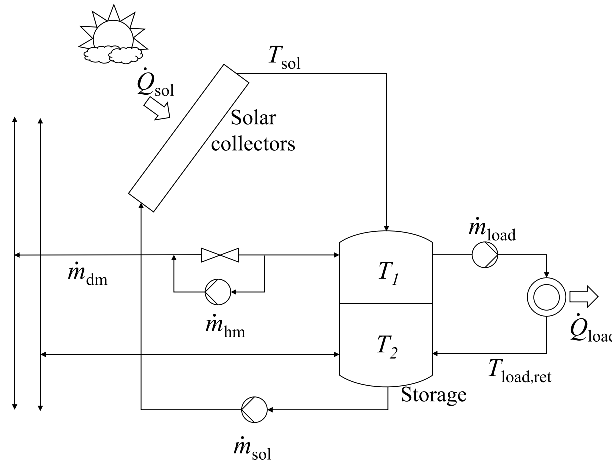

Fig. 2. Scheme of the building installations. Adapted from [12].

B. Model description

In the following the model for the simulation of the heating network consisting of N_h households (or buildings) is described in more detail. The scheme of the building installations is shown in Fig. 2. All buildings are equipped in the same way and differ only in the numbers of heat consumption $\dot{Q}_\mathrm{load}$, STC sizes and storage sizes.

The time-varying parameters heat demand profile $\dot{Q}_\mathrm{load}$ and mass flow rate $\dot{m}_\mathrm{load}$ for every household determine the respective return temperature $T_\mathrm{load,ret}$:

$$T_\mathrm{load,ret}(t) = T_1(t) - \frac{\dot{Q}_\mathrm{load}(t)}{\dot{m}_\mathrm{load}(t) c_\mathrm{p}}. \quad (1)$$

where c_p denotes the specific heat capacity of water and T_1 represents the upper storage temperature. In order to cover the heat demand, it is necessary that the uppermost storage layer T_1 does not drop below a minimum temperature $T_{1,\mathrm{min}}$ at times when there is a heat demand ($\dot{Q}_\mathrm{load}(t) > 0$).

Since the specific behavior of the collector cycle is difficult to model due to the fact that the internal controller is unknown and cannot be influenced, the solar data is calculated externally and enters the model as parameters. This mainly concerns the solar yield $\dot{Q}_\mathrm{sol}$ and the mass flow rate $\dot{m}_\mathrm{sol}$, which depend, among other factors, on the ambient temperature and solar irradiation. Both determine the STC outlet temperature T_sol, depending on the lower storage and STC inlet temperature T_2.

$$T_\mathrm{sol}(t) = T_2(t) + \frac{\dot{Q}_\mathrm{sol}(t)}{\dot{m}_\mathrm{sol}(t) c_\mathrm{p}}. \quad (2)$$

The water storage tanks are stratified storages consisting of two storage layers ($N_\mathrm{s} = 2$). The tanks are intended to further enhance the self-sufficiency of the heating network through the flexibility associated with storage options. The temperature development of the two layers of the storage tanks of household i, with water mass m_i, is determined as follows:

$$\frac{\mathrm{d}T_{i,1}(t)}{\mathrm{d}t} = \frac{1}{m_i c_\mathrm{p}}\Big(\dot{m}_{\mathrm{load},i}(t)c_\mathrm{p}\big(T_{i,2}(t) - T_{i,1}(t)\big) \tag{3a}$$

$$+ \dot{m}_{\mathrm{sol},i}(t)c_\mathrm{p}\big(T_{\mathrm{sol},i}(t) - T_{i,1}(t)\big) \tag{3b}$$

$$+ b_{\mathrm{dm},i}(t)\dot{m}_\mathrm{dm}c_\mathrm{p}\big(T_{\mathrm{dm,set}} - T_{i,1}(t)\big) \tag{3c}$$

$$+ \sum_{\substack{1\leq j\leq N_\mathrm{h}\\ j\neq i}} b_{\mathrm{hm},ij}(t)\dot{m}_\mathrm{hm}c_\mathrm{p}\big(T_{i,2}(t) - T_{i,1}(t)\big) \tag{3d}$$

$$+ \sum_{\substack{1\leq j\leq N_\mathrm{h}\\ j\neq i}} b_{\mathrm{hm},ji}(t)\dot{m}_\mathrm{hm}c_\mathrm{p}\big(T_{j,1}(t) - T_{i,1}(t)\big)\Big) \tag{3e}$$

$$\frac{\mathrm{d}T_{i,2}(t)}{\mathrm{d}t} = \frac{1}{m_i c_\mathrm{p}}\Big(\dot{m}_{\mathrm{load},i}(t)c_\mathrm{p}\big(T_{\mathrm{load,ret}} - T_{i,2}(t)\big) \tag{4a}$$

$$+ \dot{m}_{\mathrm{sol},i}c_\mathrm{p}\big(T_{i,1}(t) - T_{i,2}(t)\big) \tag{4b}$$

$$+ b_{\mathrm{dm},i}(t)\dot{m}_\mathrm{dm}c_\mathrm{p}\big(T_{i,1}(t) - T_{i,2}(t)\big) \tag{4c}$$

$$+ \sum_{\substack{1\leq j\leq N_\mathrm{h}\\ j\neq i}} b_{\mathrm{hm},ij}(t)\dot{m}_\mathrm{hm}c_\mathrm{p}\big(T_{j,2}(t) - T_{i,2}(t)\big) \tag{4d}$$

$$+ \sum_{\substack{1\leq j\leq N_\mathrm{h}\\ j\neq i}} b_{\mathrm{hm},ji}(t)\dot{m}_\mathrm{hm}c_\mathrm{p}\big(T_{i,1}(t) - T_{i,2}(t)\big)\Big) \tag{4e}$$

for $i, j = 1, ..., N_\mathrm{h}$. The upper storage temperature $T_{i,1}$ is influenced by the mass flow rate and the temperature difference of the upper and lower storage layer. The share of STC generation is represented by (3b) and (4b). Line (3c) refers to the obtained heat from the CHP plant. The binary variable b_dm indicates whether heat from the CHP plant is obtained at a point in time ($b_\mathrm{dm}(t) = 1$) or not ($b_\mathrm{dm}(t) = 0$). We furthermore assume that the CHP plant provides heat at a constant temperature $T_\mathrm{dm,set}$. Due to the mixing of the storage tank, the temperature of the lower layer changes according to (4c).

In addition to the CHP plant, buildings may also attempt to meet heating demands through the preferred option in the optimization via appropriate individual exchanges among themselves. The binary variable $b_{\mathrm{hm},ij}$ determines whether there is a heat flow from building i to building j ($b_{\mathrm{hm},ij}(t) = 1$). All heat releases taking place from building i to one or multiple buildings j and the return flows are summed up, cf. (3d), (4d). The same holds for building i receiving heat from one or multiple other buildings j, cf. (3e), (4e). The mass flow rate between the buildings $\dot{m}_\mathrm{hm}$ and from the CHP plant $\dot{m}_\mathrm{dm}$ are assumed to be constant. We note that the Ordinary Differential Equation (ODE) is linear in states T and that we only have binary controls. This leads to the special structure of (9b) as shown later.

III. Optimal control formulation of the heating network

In this section we formulate the MIOCP of the model introduced in Section II, followed by the two approaches to solve the optimization problem exploiting its special form.

A. Optimal control problem

The goal of the optimization is to minimize the requested heat from the CHP plant. According to the problem definition, the CHP plant should be active only when the heat demand of the buildings exceeds the stored amount and generated heat. This could be achieved, for example, through predetermined time slots, such as times when there is no STC output. This would, however, restrict the optimization to a greater extent, and we instead leave the decision of the activation of the CHP plant to the solver. In order to determine the activation of the CHP plant with only one variable, we introduce an auxiliary binary variable b_cp for the on/off status of the CHP plant, making use of a so-called big-M constraint:

$$\sum_{i=1}^{N_\mathrm{h}} b_{\mathrm{dm},i}(t) \leq M \cdot b_\mathrm{cp}(t). \tag{5}$$

Otherwise, we would have to check every single binary variable of each household for sourcing heat from the CHP plant (b_dm). This constraint results in the CHP plant being on ($b_\mathrm{cp}(t) = 1$) if at least one building obtains heat from the CHP plant, otherwise the CHP plant remains off and accordingly $b_\mathrm{cp}(t) = 0$. The value of M is chosen large enough so that b_cp becomes 1 if the CHP plant must be activated.

While we have fairly little constraints for activating the CHP plant, there are several conceivable ways to restrict activation for the household exchange. We may allow only one transfer between two households at each point in time. However, this is a very strong restriction and may lead to an increased use of the CHP plant, which is not desirable. Then again, we could impose no constraints at all and, similarly to the CHP plant, allow all buildings to trade with each other at any time when the household mode is active. In real-life applications, this may not be desirable either, especially when the two exchanging households are not in close neighborhood and the heat exchange over longer distances would lead to significant heat losses. This becomes more relevant the larger the network is.

Within this work, we impose that the buildings can transfer heat in multiple 1:1-relationships at a point in time.

$$\sum_{i=1}^{N_\mathrm{h}} b_{\mathrm{hm},ij}(t) \leq 1, \quad j = 1, .., N_\mathrm{h}, i \neq j. \tag{6}$$

Moreover, we have to exclude via a constraint that two buildings do not send heat quantities back and forth to each other, since our model formulation would not prevent that.

$$b_{\mathrm{hm},ij}(t) + b_{\mathrm{hm},ji}(t) \leq 1, \quad i, j = 1, ..., N_\mathrm{h}, i \neq j. \tag{7}$$

Using the auxiliary binary variable b_cp for the CHP plant (5), we make sure that both the household mode (HM) and the CHP plant (DM) are not operated at the same time:

$$b_{\mathrm{hm},ij}(t) \leq 1 - b_\mathrm{cp}(t), \quad i, j = 1, ..., N_\mathrm{h}, i \neq j. \tag{8}$$

Let $x \in \mathbb{R}^{n_x}$ be the differential states of a nonlinear system, $u \in \mathbb{R}^{n_u}$ the continuous controls, $b \in \{0, 1\}^{n_b}$

the binary controls, $c \in \mathbb{R}^{n_c}$ time-varying parameters, and $s \in \mathbb{R}^{n_s}$ a number of slack variables with $n_x, n_u, n_c, n_s \in \mathbb{N}$.

The number of differential states for the residential heating network depends on the number of buildings. The model contains $n_x = N_\mathrm{s} \cdot N_\mathrm{h}$ states, with N_s being the number of storage layers and N_h the number of households, i.e. for two storage layers ($N_\mathrm{s} = 2$) and three households ($N_\mathrm{h} = 3$) we get $n_x = 6$. We therefore have for the state vector $x^\top = [T_{1,1}\ T_{1,2}\ T_{2,1}\ T_{2,2}\ T_{3,1}\ T_{3,2}]$. The amount of decision variables solely depends on the number of buildings. For the case of $N_\mathrm{h} = 3$ households, we have a total of $n_b = 10$ binary controls, since every building has one variable for the CHP plant ($b_{\mathrm{dm},i}(t)$), two variables for the household exchange possibilities, e.g. building 1 can transfer heat to building 2 and building 3, and the additional auxiliary variable b_cp. The decision variable vector is hence defined as $b^\top = [b_\mathrm{cp} b_{\mathrm{dm},1} b_{\mathrm{dm},2} b_{\mathrm{dm},3} b_{\mathrm{hm},12} b_{\mathrm{hm},13} b_{\mathrm{hm},21} b_{\mathrm{hm},23} b_{\mathrm{hm},31} b_{\mathrm{hm},32}]$. Note that the model consists of only binary controls. The number of time-varying parameters is $n_c = 4N_\mathrm{h}$, namely $c^\top = [\dot{Q}_{\mathrm{load},\{1\ldots N_\mathrm{h}\}}, \dot{m}_{\mathrm{load},\{1\ldots N_\mathrm{h}\}}, \dot{Q}_{\mathrm{sol},\{1\ldots N_\mathrm{h}\}}, \dot{m}_{\mathrm{sol},\{1\ldots N_\mathrm{h}\}}]$. Following the model description and the listed constraints, we obtain the MIOCP for the heating network as:

$$\min_{\substack{x(\cdot),b(\cdot),\\ s(\cdot)}} \quad \int_{t_0}^{t_f} \big(c_\mathrm{b}^\top b(t) + c_\mathrm{s}^\top s(t)\big)\mathrm{d}t \tag{9a}$$

$$\text{s.t.} \quad \text{for } t \in [t_0, t_f]:$$

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = A_0 x(t) + E_0 c(t) + \sum_{i=1}^{n_b} b_i(t)\big(A_i x(t) + E_i c(t)\big), \tag{9b}$$

$$(5) - (8), \tag{9c}$$

$$T_\mathrm{lb} - s(t) \leq x(t) \leq T_\mathrm{ub} + s(t), \tag{9d}$$

$$b(t) \in \{0,1\}^{n_b}, \tag{9e}$$

$$s(t) \geq 0, \tag{9f}$$

$$x(t_0) = \bar{x}_0. \tag{9g}$$

The objective minimizes the sum of all binary variables, as well as the slack variables, both weighted by a corresponding cost vector $c_\mathrm{b} \in \mathbb{R}^{n_b}$ and $c_\mathrm{s} \in \mathbb{R}^{n_s}$, with $n_s = n_x$. The weighting vector c_b is chosen to the extent that the activation of the CHP plant is priced more highly than the household exchange. The system dynamics introduced in equations (3) and (4) can be expressed as a switched system, shown in (9b). Equation (9c) refers to the conditional constraints (5)-(8). Bounds on the states, augmented by slack variables, are given by (9d). Equation (9e) represent the binary constraint, while (9f) restricts the value of the slack variables to be positive and (9g) determines the initial state.

For the solution of the above optimization problem, we compare two methods exploiting the special form of the system dynamics. Since problem (9) contains an infinite number of values in the time horizon $t \in [t_0, t_f]$, the infinite-dimensional MIOCP has to be discretized at some point [13]. We use for both methods the direct multiple shooting approach [14] incorporating a 2-step implicit Euler method.

B. Combinatorial Integral Approximation

In order to solve the MINLP arising from the discretization, we make use of the so-called Combinatorial Integral Approximation (CIA) method [6], [11]. The method, as stated in [15], is summarized in Algorithm 1. The idea of CIA is to solve discretized MIOCPs by decomposing the original problem into a sequence of three subproblems. First, a relaxed MINLP, where the binary variables are relaxed ($b_\mathrm{rel} \in [0,1]^{n_b}$) and thus a NLP ($\mathrm{NLP_{rel}}$) is solved. Using the solution of $\mathrm{NLP_{rel}}$, the so-called CIA problem [11] is solved to yield approximations of the binary controls $b_\mathrm{bin} \in \{0,1\}^{n_b}$. In the third step, the NLP is solved again with the approximated binary controls fixed $b = b_\mathrm{bin}$, to adjust the remaining optimization variables such as states, continuous controls, and (if necessary) slack variables to the obtained binary solution. Bounds for the rounding error are supported by a theorem by Sager et al. [16]. With regard to real-world applications and the use of model predictive control, the slack variables have an even greater importance as these can be used to soften the path constraints to ensure a feasible solution [13].

Algorithm 1: CIA decomposition algorithm for solution of MIOCPs

Input : Discretized MIOCP (MINLP), initial guesses for x, u, b.
Output: Local optimal variables x^*, u^*, b^*, s^* with objective $F^* = F(x^*, u^*, b^*, s^*)$.

1 Solve relaxed MINLP ($\mathrm{NLP_{rel}}$) $\to x, u, b_\mathrm{rel}, F$;
2 Solve CIA problem for $b_\mathrm{rel} \to b_\mathrm{bin}$;
3 Solve MINLP with $b = b_\mathrm{bin}$ fixed ($\mathrm{NLP_{bin}}$) $\to x, u, F_\mathrm{bin}$;
4 **return** $(x^*, u^*, b^*, s^*, F^*) = (x, u, b_\mathrm{bin}, F_\mathrm{bin})$;

In the problem formulation used in this work, only binary controls are considered, so that only states and slack variables are remaining as optimization variables in step 3. In this special case, the final NLP solution step could also be replaced by a numerical simulation of the ODE system (9b) using the initial state $\bar{x}_0$ and the binary controls b_bin.

The CIA problem (step 2) itself is a Mixed-Integer Linear Program (MILP) and defined as:

$$\min_{b_\mathrm{bin},\theta} \quad \theta \tag{10a}$$

$$\text{s.t.} \quad \text{for } k = 0, ..., N-1:$$

$$\theta \geq \pm \sum_{j=0}^{k} (b_{\mathrm{rel},\{j,i\}} - b_{\mathrm{bin},\{j,i\}}) \cdot (t_{j+1} - t_j), \quad i = 1, ..., n_b, \tag{10b}$$

$$1 \geq \sum_{i=1}^{n_b} b_{\mathrm{bin},\{k,i\}}, \tag{10c}$$

$$b_{\mathrm{bin},k} \in \{0,1\}^{n_b}. \tag{10d}$$

The main idea of CIA is to minimize the maximum integrated difference θ between the relaxed b_{rel} and binary controls b_{bin} as defined by (10b). Constraint (10c) ensures that at most one of the considered binary controls can be active at a time. The CIA problem (10) can be solved, e.g., using standard MILP solvers or tailored Branch-and-Bound (BnB) algorithms [11]. In this work, the Python module pycombina [17] is used to solve CIA problems. For this application, we solve multiple CIA problems, one for each set of mutually exclusive binary variables. The NLPs are implemented in CasADi v3.5.5 [18] using Python and solved using Ipopt [19] with linear solver MA57 [20].

C. Discretized Linear Reformulation

As MINLPs are generally hard to solve, we have resorted to an approximate solution, using the CIA method. Yet, the results of the CIA method do not guarantee an optimal solution. In order to exploit optimality and the use of linear programs, we introduce a Discretized Linear Reformulation (DLR) of the problem (9) in the following. This mainly concerns the reformulation of the non-linearity of our model, consisting of the product of a binary control b and a continuous state x.

Assuming that we have n_b binary controls b_i, each dynamic system is linear in states x, and the overall set of admissible states $\mathbb{X}$ is compact and nonempty. Then one can introduce additional algebraic controls z_i, such that an equivalent system equation that is linear in z_i with convex constraints coupling z_i and x is obtained [21]. We recall the switched system formulation of the system dynamics:

$$\begin{aligned}\frac{\mathrm{d}x(t)}{\mathrm{d}t} &= A_0 x(t) + E_0 c(t) \\ &\quad + \sum_{i=1}^{n_b} b_i(t)\big(A_i x(t) + E_i c(t)\big)\end{aligned} \tag{11}$$

with $x \in \mathbb{X}$ and $\mathbb{X}$ being a compact set. Defining the auxiliary variables z_i as:

$$z_i(t) := b_i(t)x(t), \tag{12}$$

leads to the system equation being reformulated as:

$$\begin{aligned}\frac{\mathrm{d}x(t)}{\mathrm{d}t} &= A_0 x(t) + E_0 c(t) \\ &\quad + \sum_{i=1}^{n_b} \big(A_i z_i(t) + b_i(t) E_i c(t)\big).\end{aligned} \tag{13}$$

This then allows us to impose the following linear conditions on the auxiliary variable z:

$$x(t) - z_i(t) \in \big(1 - b_i(t)\big) \cdot \mathbb{X}, \tag{14a}$$

$$z_i(t) \in b_i(t) \cdot \mathbb{X}. \tag{14b}$$

Using this reformulation approach and applying the implicit Euler integration scheme with time step $\Delta t = \frac{t_f - t_0}{N}$ results in the following discretized linear problem formulation:

$$\min_{\substack{x,z,\\ b,s}} \sum_{k=0}^{N-1} \Delta t (c_{\text{b}}^\top b_k + c_{\text{s}}^\top s_k) + c_{\text{s}}^\top s_N \tag{15a}$$

$$\text{s.t.} \quad \begin{aligned} x_{k,1} &= x_{k,0} + \frac{\Delta t}{2}\big(A_0 x_{k,1} + E_0 c_k \\ &\quad + \sum_{i=1}^{n_b} (A_i z_{k,i,0} + b_{k,i} E_i c_k)\big), k = 0, ..., N-1, \\ x_{k+1,0} &= x_{k,1} + \frac{\Delta t}{2}\big(A_0 x_{k+1,0} + E_0 c_k \\ &\quad + \sum_{i=1}^{n_b} (A_i z_{k,i,1} + b_{k,i} E_i c_k)\big), k = 0, ..., N-1, \end{aligned} \tag{15b}$$

$$(5) - (8), \quad k = 0, ...N-1, \tag{15c}$$

$$T_{\text{lb}} - s_k \le x_k \le T_{\text{ub}} + s_k, \quad k = 0, ..., N, \tag{15d}$$

$$b_k \in \{0,1\}^{n_b}, \quad k = 0, ..., N-1, \tag{15e}$$

$$s_k \ge 0, \quad k = 0, ..., N, \tag{15f}$$

$$\begin{aligned}(1 - b_k)T_{\text{lb}} \le x_k - z_k \le (1 - b_k)T_{\text{ub}}, \\ k = 0, ..., N-1,\end{aligned} \tag{15g}$$

$$b_k T_{\text{lb}} \le z_k \le b_k T_{\text{ub}}, \quad k = 0, ..., N-1, \tag{15h}$$

$$x_0 = \bar{x}_0. \tag{15i}$$

The DLR approach is also implemented in CasADi v3.5.5 [18] using Python. The MILP is solved using Gurobi 9.1.2 with default settings via CasADi [22]. All computations were performed on an Intel(R) Core(TM) i5-10310U 1.70GHz CPU and 16GB RAM running Ubuntu 20.04.

IV. Comparison of Results

In this section we present the solution of the MINLP using the Combinatorial Integral Approximation method (CIA-MINLP) and the solution of the MILP resulting from the Discretized Linear Reformulation (DLR-MILP).

We tested both methods on a variety of heating network configurations. This mainly includes the optimization for different amount of buildings within the network. The heat demands, storage sizes and solar yield are adapted depending on the number of participants in the network. However, to ensure comparability the only difference between the MINLP and MILP is the reformulation of the nonlinearities. The respective heat demands over the period of the optimization horizon (24 hours) for $N_{\text{h}} = 4$ are shown in Fig. 3.

It is assumed that most buildings are mainly home to households with their specific heat patterns of an early summer day. This is indicated by the orange curve ($\dot{Q}_{\text{load},2}$) and the dashed red curve ($\dot{Q}_{\text{load},4}$). The exemplary heating demand of building 2 and 4 correspond to an average sized household determined by the ambient temperature and hence more likely to be higher during nighttimes and very low during the day as shown in the graph. We furthermore assume that there are some occasional facilities that require a rather constant and permanent heat demand throughout the day ($\dot{Q}_{\text{load},1}$) and buildings that have more heat demand during the day ($\dot{Q}_{\text{load},3}$). Each optimization period is one day (24

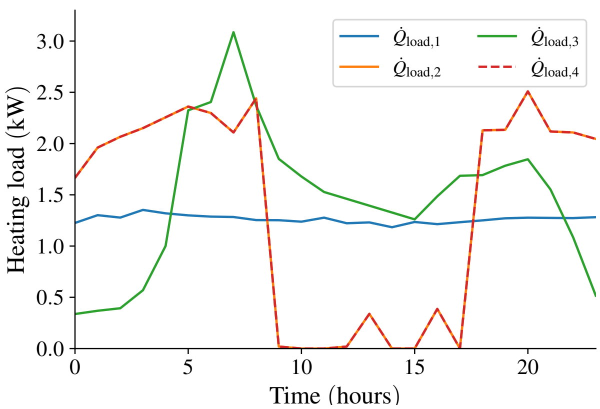

Fig. 3. Exemplary heat demand of actors of the residential heating network.

hours), starting at midnight, with $N = 24$ equidistant time points and hence $\Delta t = 1\,\mathrm{h}$.

The results for both methods are depicted in Fig. 4. Two plots are presented for each building. The upper plots show the temperature evolution in the hot water tank, while the lower plots represent the values of the binary control variables. The arrows in the legend regarding the binary control variables indicate the direction of heat transfer, e.g. $b_{1\to 2}$ means that building 1 transfers heat to building 2, while b_{dm} means that the respective building draws heat from the CHP plant. The results obtained using CIA are shown as solid curves, and the dotted lines correspond to the solution from the DLR approach. The figure shows that both methods lead to identical system behavior, at least until the seventh hour of the optimization. For both methods, all buildings have to draw heat from the CHP plant starting at the beginning of the horizon until 2 o'clock and again another two hours after a short break with no heat exchange of 1 hour duration. The first deviating behavior between the two methods is, when building 2 supplies heat to building 3 from 7 to 8 o'clock for the CIA case, which is not done for the DLR case. The different actions of the buildings can also be seen in the clearly deviating temperature development of building 4 starting at around 13 o'clock.

A comparison of key figures for both methods is recorded in Table I. It is noticeable that for all scenarios the objective value obtained by CIA is notably higher than the one from DLR. This is easily explainable, though. Since the CIA method has to exploit the purpose of slack variables, the objective value is significantly higher due to the high pricing of slack variables. The DLR-MILP on the other side does not utilize any slack variables. However, further interesting numbers are the computation time and the drawn heat from the CHP plant ($\sum_{i=1}^{N_{\mathrm{h}}} \sum_{k=1}^{N} Q_{\mathrm{dm}}$). For the computation time, we can state that CIA has a clear advantage over DLR. Please note, that we interrupt the Gurobi solver after a time limit of 600 seconds and specify the calculated solution and lower bound until then. It could be that the latest found integer solution is already optimal, and the lower bound

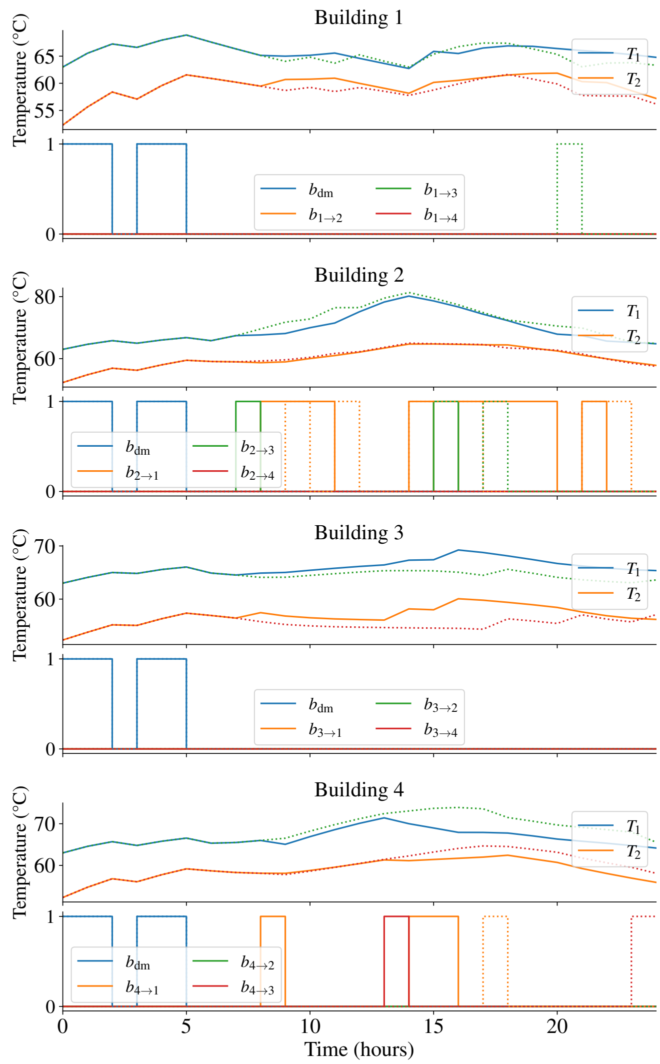

Fig. 4. Binary solution of CIA-MINLP (solid lines) and DLR-MILP (dotted lines) for $N_{\mathrm{h}} = 4$.

would further converge towards the upper bound. Regarding the total drawn heat from the CHP plant, the CIA method performs slightly worse. However, the difference is nearly negligible with deviations of around $1 - 4\,\%$ (for up to $N_{\mathrm{h}} = 5$). The deviation of the computation time clearly outweighs the marginal difference of the drawn heat.

Once the network size reaches six households, the DLR-MILP approach struggles to find a feasible solution within our set time limit of 600 seconds. This may be mainly caused by the significant increase of the problem size compared to the CIA-MINLP approach. While the number of states n_x has a linear relationship with the number of households and storage layers, the number of binary variables increases quadratically with every further household. The number of z variables for the linear reformulation approach is even expressible as a cubic function. Thus, the DLR-MILP gets much larger with an increase of households. In numerical terms, this means for the case of $N_{\mathrm{h}} = 5$, or $N_{\mathrm{h}} = 6$ households, an increase in the number of states from $n_{x,5} =$

TABLE I

COMPARISON OF SELECTED KEY FIGURES

N_{h}	Objective value			Computation time (s)		$\sum_{i=1}^{N_{\mathrm{h}}} \sum_{k=1}^{N} Q_{\mathrm{dm}}$ (kWh)		$\max_{k=1,\ldots,N} s_k$ (°C)	
	CIA-MINLP	DLR-MILP	Lower Bound	CIA-MINLP	DLR-MILP	CIA-MINLP	DLR-MILP	CIA-MINLP	DLR-MILP
2	1.15e+03	3.20e+02	3.20e+02	0.9	9.9	17.9	17.3	0.36	0.0
3	1.00e+03	3.60e+02	3.10e+02	1.6	600	21.6	21.0	0.45	0.0
4	1.17e+03	4.70e+02	2.10e+02	3.1	600	33.5	33.0	0.57	0.0
5	2.40e+03	5.70e+02	1.90e+02	9.2	600	43.9	42.5	0.40	0.0
6	1.70e+03	-	1.90e+02	11.1	600	54.0	-	0.52	-

10 to $n_{x,6} = 12$, an increase in the number of binary control variables from $n_{b,5} = 25$ to $n_{b,6} = 36$ and an increase in the number of z variables from $n_{z,5} = 90$ to $n_{z,6} = 132$.

V. CONCLUSION

Within this work we examined and compared the results of two methods for optimal control of switched linear systems applied to a residential heating network. Keeping the system-related nonlinearities and solving the resulting MINLP via the CIA method leads to slightly worse key figures, but a shorter computation time compared to the DLR-MILP. We showed that using the linear reformulation does not give a real advantage for this application. The much higher computation time with only minimal better values is, especially with regard to possible MPC applications a clear disadvantage. The need of z variables and further constraints leads to an increased problem size compared to CIA-MINLP and makes it hard to find a feasible solution for networks of more than five households within a reasonable time.

Future work could therefore include trying to speed up the DLR-MILP. Providing feasible warm-starts and careful tuning of solver settings may result in faster computation.

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