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First-Order Sweeping Processes and Extended Projected Dynamical Systems: Equivalence, Time-Discretization and Numerical Optimal Control

Anton Pozharskiy[1], Armin Nurkanović[1], Moritz Diehl[1,2]

***Abstract*— Constrained dynamical systems are systems such that, by some means, the state stays within a given set. Two such systems are the (perturbed) Moreau sweeping process and the recently proposed extended Projected Dynamical System (ePDS) [1]. We show that under certain conditions solutions to the ePDS correspond to the solutions of a dynamic complementarity system, similar to the one equivalent to ordinary PDS. We then show that the perturbed sweeping process with time varying set can, under similar conditions, be reformulated as an ePDS. In this paper, we leverage these equivalences to develop an accurate discretization method for perturbed first-order Moreau sweeping processes via the finite elements with switch detection method [2]. This allows the efficient optimal control of systems governed by ePDS and perturbed first-order sweeping processes.**

I. INTRODUCTION

In recent years there has been renewed interest in nonsmooth dynamical systems in general and particularly in Projected Dynamical Systems (PDS) and sweeping processes. When PDS were first introduced by Henry [3], and more recently studied by Dupuis and Nagurney [4], they involved smooth ODEs constrained to a set $\mathcal{C}$. The requirements on the coupled set has since then been relaxed to simply ones where the tangent cone is convex, i.e. the set is uniformly prox-regular. This class of systems have been used to study various systems including vaccination strategies [5] and economic problems [6]. There has also been recent work connecting PDS with the popular field of control barrier functions [7]. The resurgence of interest has lead to several extensions to PDS, namely so called “Extended” and “Oblique” Projected Dynamical Systems (ePDS and oPDS [8], respectively). Both extensions modify the projection operator used to transform the dynamics such that the state stays within $\mathcal{C}$, and have been used to treat novel control systems [1], [9].

First-order Sweeping Processes (FOSwP) were introduced originally by Moreau [10]. Similarly to PDS, FOSwP have seen recent work in an optimal control context such as in the case of marine surface vehicles [11] and crowd motion [12]. FOSwP have also been applied in soft robotics contexts [13]. Recently, a numerical method using measure relaxations has been proposed, though only for stationary sets $\mathcal{C}$ [14].

We have previously introduced an accurate discretization method for ordinary PDS using the Finite Elements with

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[1]Department of Microsystems Engineering (IMTEK), [2]Department of Mathematics, University of Freiburg, Germany, {anton.pozharskiy,armin.nurkanovic,moritz.diehl} @imtek.uni-freiburg.de

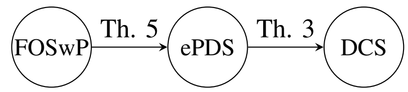
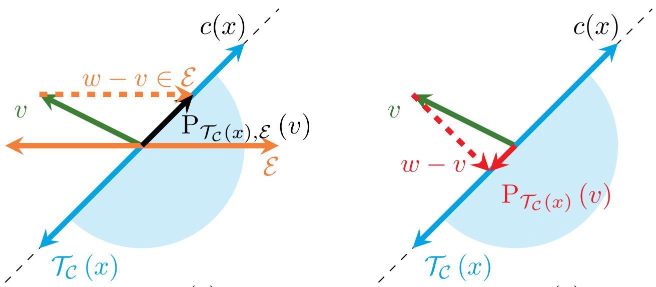


Fig. 1: Relationships between systems in this paper with edge labels denoting the theorems that relate them.

Switch Detection (FESD) method [15]. Due to the equivalence of PDS with FOSwP with time invariant sets (sometimes called reflecting boundaries), the systems handled in [11], [12], could also accurately be discretized using this method. However, if the sweeping sets are time dependent then this equivalence does not hold. This limitation precludes the discretization of the sweeping processes with time varying sets such as the one in [16, Section 8.1]. To address this we look to ePDS as we require a similar remedy as in [1], introducing a clock state the dynamics of which are not projected.

Contributions: This paper proves that under some assumptions on the time dependence of the sweeping set $\mathcal{C}(t)$, FOSwP can be reformulated into an ePDS. These ePDS then can be shown to be equivalent to a dynamic complementarity system which can be accurately discretized via a slight modification of the FESD discretization for PDS [15]. We then provide several examples of optimal control problems with this class of system. These examples and the discretization method are implemented and available in the open-source software package `nosnoc` [17].

Notation: Regard a closed set $\mathcal{C} \subseteq \mathbb{R}^n$. The tangent cone to $\mathcal{C}$ at x, is defined as $\mathcal{T}_{\mathcal{C}}(x) = \{\, d \in \mathbb{R}^n \mid \exists \{x_i\} \in \mathcal{C}, \{t_i\} \in \mathbb{R}_+, x_i \to x, t_i \to t, d = \lim_{i\to\infty} \frac{x_i - x}{t_i} \,\}$. The normal cone to $\mathcal{C}$ at x is defined by $\mathcal{N}_{\mathcal{C}}(x) = \{\, v \in \mathbb{R}^n \mid \langle v, d\rangle \leq 0, \forall d \in \mathcal{T}_C(x) \,\}$, for $x \in \mathcal{C}$. In the interior of $\mathcal{C}$, $\mathcal{T}_{\mathcal{C}}(x) = \mathbb{R}^n$ and $\mathcal{N}_{\mathcal{C}}(x) = \{0\}$, and both are the empty set outside $\mathcal{C}$. A finitely defined $\mathcal{C} \subset \mathbb{R}^n$ is written as $\mathcal{C} = \{\, x \in \mathbb{R}^n \mid c(x) \geq 0 \,\}$ with $c : \mathbb{R}^n \to \mathbb{R}^{n_c}$. In the sequel, we also use $\mathcal{A}(x)$ to denote the index set of active constraints: $\mathcal{A}(x) = \{\, i \in \{1, \dots, n_c\} \mid c_i(x) = 0 \,\}$. We also define $\nabla c_{\mathcal{A}}(x)$ as $\nabla c(x)$ with only the columns corresponding to the indices in $\mathcal{A}(x)$. Let $\mathcal{K} \subseteq \mathbb{R}^n$ be a closed convex set. The Euclidean projection operator is defined via a convex optimization problem $\mathrm{P}_{\mathcal{K}}(v) = \arg\min_{w\in\mathcal{K}} \frac{1}{2}\|w - v\|_2^2$. Let linear subspace $\mathcal{E}$ be the space spanned by the columns of the matrix $E \in \mathbb{R}^{n\times n'}$, with $n' \leq n$, i.e., $\mathcal{E} = \{\, s \in \mathbb{R}^n \mid EE^\top x = s,\ x \in \mathbb{R}^n \,\} \subseteq \mathbb{R}^n$. We denote $\mathcal{E}_\perp = \mathbb{R}^n/\mathcal{E}$ to be the orthogonal complement of $\mathcal{E}$ which is itself a linear subspace. Projection of a vector $v \in \mathbb{R}^n$ onto the linear subspace $\mathcal{E}$ has a closed form solution, namely, $\mathrm{P}_{\mathcal{E}}(v) = EE^\top v$, if the columns of E form an orthonormal basis of $\mathcal{E}$. In the following we use $0_n \in \mathbb{R}^n$ to define a



(a) $\mathrm{P}_{\mathcal{T}_\mathcal{C}(x),\mathcal{E}}\left(v\right)$. (b) $\mathrm{P}_{\mathcal{T}_\mathcal{C}(x)}\left(v\right)$.

Fig. 2: Comparing the extended projection operator (left) and the Euclidean projection operator (right).

column vector of n zeros.

II. Background

In this section, we discuss some of the mathematical background required for the remainder of the paper. This first covers the properties of the ePDS and the projection operator it uses, which is followed by an overview of the perturbed first-order Moreau sweeping process.

A. Extended PDS

The extended projection operator $\mathrm{P}_{\mathcal{K},\mathcal{E}}\left(f(x,u)\right)$ is defined as:

$$\mathrm{P}_{\mathcal{K},\mathcal{E}}\left(v\right) = \underset{w\in\mathbb{R}^n}{\arg\min}\ \frac{1}{2}\|w-v\|_2^2, \tag{1a}$$

$$\text{s.t.} \quad w \in \mathcal{K},\ w - v \in \mathcal{E} \tag{1b}$$

This convex program is, intuitively, equivalent to finding the closest point $w \in \mathcal{K}$ to v which can be reached via a straight line in the subspace $\mathcal{E}$. An example of the difference between the standard Euclidean projection and this operator can be seen in fig. 2. If we apply this projection operator with $\mathcal{K} = \mathcal{T}_\mathcal{C}\left(x\right)$ and $v = f(x,u)$, the resulting program is necessarily convex if the tangent cone $\mathcal{T}_\mathcal{C}\left(x\right)$ is convex and its feasible set, $\mathcal{T}_C\left(x\right) \cap f(x,u) + \mathcal{E} \neq \emptyset$. This yields the extended PDS, introduced in [1], which takes the form:

$$\dot{x} = \mathrm{P}_{\mathcal{T}_\mathcal{C}(x),\mathcal{E}}\left(f(x,u)\right), \tag{2a}$$

$$x(0) \in \mathcal{C}, \tag{2b}$$

with time-independent $\mathcal{C} = \{\, x \in \mathbb{R}^n \mid c(x) \geq 0 \,\}$ and $\mathcal{E}$ which is a linear subspace of $\mathbb{R}^n$. In the context of these systems the nonemptiness of the feasible set in eq. (1) is also useful in conjunction with [18, Theorem 2.4] to guarantee that the state x stays in $\mathcal{C}$ for all $t \in [0,T]$ and therefore solutions exist on the interval $[0,T]$.

B. First-Order Sweeping Processes

We treat the perturbed first-order sweeping process:

$$\dot{x} \in f(x,u) - \mathcal{N}_{\mathcal{C}(t)}\left(x\right), \tag{3}$$

with the time-dependent set $\mathcal{C}(t) = \{\, x \in \mathbb{R}^n \mid c(x,t) \geq 0 \,\}$, defined by the function $c : \mathbb{R}^{n_x} \times \mathbb{R} \to \mathbb{R}^{n_c}$. We assume $f : \mathbb{R}^{n_x} \times \mathbb{R}^{n_u} \to \mathbb{R}^{n_x}$ is at least once differentiable in both x and u throughout this paper.

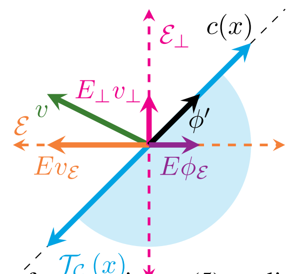


Fig. 3: Schematic of vectors in eq. (5) applied to $\mathcal{K} = \mathcal{T}_\mathcal{C}\left(x\right)$.

III. Equivalence of ePDS to a DCS

In this section, we discuss the equivalence of the ePDS to particular Dynamic Complementarity System (DCS). The set $\mathcal{C}$ is considered to be time-independent and finitely defined by $c : \mathbb{R}^{n_x} \to \mathbb{R}^{n_c}$.

A. Dynamic Complementarity System

We now prove that the ePDS can be reformulated into DCS. For this we make the following assumption:

Assumption 1. *For linear subspace $\mathcal{E} = \operatorname{span} E \subseteq \mathbb{R}^n$, E and $E_\perp$ are both selection matrices, i.e. $[EE_\perp] = I \in \mathbb{R}^{n\times n}$, and form an orthogonal decomposition of the space $\mathbb{R}^n$. The columns of $E \in \mathbb{R}^{n\times n'}$ form orthonormal basis for the subspace $\mathcal{E}$ and the columns of $E_\perp \in \mathbb{R}^{n\times(n-n')}$ form an orthonormal basis for the orthogonal subspace $\mathcal{E}_\perp$.*

In preparation for that we prove the following lemma for an equivalent projection definition of $\mathrm{P}_{\mathcal{K},\mathcal{E}}\left(v\right)$:

Lemma 2. *Given Assumption 1, the subspace projection operator onto convex cone $\mathcal{K} \subset \mathbb{R}^n$ with $\mathcal{E} = \operatorname{span} E \subseteq \mathbb{R}^n$:*

$$\phi = \mathrm{P}_{\mathcal{K},\mathcal{E}}\left(v\right) = \underset{w\in\mathbb{R}^n}{\arg\min}\ \frac{1}{2}\|w-v\|_2^2, \tag{4a}$$

$$\textit{s.t. } w \in \mathcal{K},\ w - v \in \mathcal{E} \tag{4b}$$

is equivalent to the projection in the subspace (i.e. $\phi = \phi'$):

$$v_\mathcal{E} = E^\top v,\ v_\perp = {E_\perp}^\top v, \tag{5a}$$

$$\phi' = E\phi_\mathcal{E} + E_\perp v_\perp, \tag{5b}$$

$$\phi_\mathcal{E} = \underset{\tilde{w}\in\mathbb{R}^{n'}}{\arg\min}\ \frac{1}{2}\|\tilde{w}-v_\mathcal{E}\|_2^2, \tag{5c}$$

$$\textit{s.t. } E\tilde{w} + E_\perp v_\perp \in \mathcal{K} \tag{5d}$$

Proof. Note that both forms of the projection operator, eq. (4) and eq. (5), are strictly convex and have the same feasible set. This comes directly from the fact that $E\phi_\mathcal{E} + {E_\perp}^\top v_\perp - v = (\phi_\mathcal{E} - v_\mathcal{E}, 0) \in \mathcal{E}$ and the constraint in eq. (5d). Now we show that $\phi = \phi'$ for all $\mathcal{E}$, $\mathcal{K}$, and v. Assume for contradiction that $\phi_\mathcal{E} \neq E^\top\phi$. If $\|\phi - v\|_2^2 < \|\phi_\mathcal{E} - v_\mathcal{E}\|_2^2$ then $\left\|E^\top\phi - v_\mathcal{E}\right\|_2^2 < \|\phi_\mathcal{E} - v_\mathcal{E}\|_2^2$ which contradicts eq. (5c). If $\|\phi - v\|_2^2 > \|\phi_\mathcal{E} - v_\mathcal{E}\|_2^2$ then $\|(\phi_\mathcal{E}, v_\perp) - v\|_2^2 < \|\phi - v\|_2^2$ which contradicts eq. (4a). This follows from the fact that the orthogonal component of ϕ, must be exactly $E^\top v$ and therefore has a residual of zero in eq. (4a). Therefore, $\phi = \phi'$. □

Theorem 3. *Assume that $\mathcal{C}$ is a finitely defined set: $\mathcal{C} = \{ x \in \mathbb{R}^{n_x} \mid c(x) \geq 0 \}$, and $c_i(x)$ for $i \in \mathcal{A}(x)$ satisfy the Linear Independence Constraint Qualification (LICQ) for all $x(t)$, i.e., $\nabla c_i(x)$ are linearly independent for $i \in \mathcal{A}(x)$. Further, assume that given control signal $u : [0,T] \to \mathbb{R}^{n_u}$ and corresponding solution $x : [0,T] \to \mathbb{R}^{n_x}$ to 2, $\mathcal{T}_C(x) \cap f(x,u) + \mathcal{E} \neq \emptyset$, for all $t \in [0,T]$. The absolutely continuous solutions $x(t)$ to the ePDS in eq.* (2) *with $\mathcal{E} = \operatorname{span} E \subseteq \mathbb{R}^{n_x}$ and $x(0) \in \mathcal{C}$, are equivalent to solutions to the DCS:*

$$\dot{x} = f(x,u) + EE^\top \nabla_x c(x)\lambda, \tag{6a}$$

$$0 \leq \lambda \perp c(x) \geq 0. \tag{6b}$$

Proof. We apply the same simplifying assumptions as in lemma 2 to define $E \in \mathbb{R}^{n_x \times n'}$ and $E_\perp \in \mathbb{R}^{n_x \times (n_x - n')}$. Recall that the tangent cone to $\mathcal{C}$ at x with active set $\mathcal{A}(x)$ is $\mathcal{T}_\mathcal{C}(x) = \{ v \in \mathbb{R}^n \mid \nabla c_i(x)^\top v \geq 0, i \in \mathcal{A}(x) \}$. This cone is closed and convex if the constraint functions $c_i(x),\ i \in \mathcal{A}(x)$ satisfy a constraint qualification such as LICQ [19]. Now we use the equivalent optimization problem in eq. (5) for the right hand side of eq. (2), i.e:

$$\begin{aligned}
\dot{x} &= E\dot{x}_\mathcal{E} + E_\perp v_\perp, \\
v_\mathcal{E} &= E^\top f(x,u),\ v_\perp = {E_\perp}^\top f(x,u), \\
\dot{x}_\mathcal{E} &= \underset{\tilde{w} \in \mathbb{R}^{n'}}{\arg\min} \|\tilde{w} - v_\mathcal{E}\|, \\
&\qquad \text{s.t. } E\tilde{w} + E_\perp v_\perp \in \mathcal{T}_\mathcal{C}(x).
\end{aligned}$$

Substituting $v = E\tilde{w} + E_\perp v_\perp$ into the definition of $\mathcal{T}_\mathcal{C}(x)$ results in the polyhedral set:

$$\{ \tilde{w} \in \mathbb{R}^{n'} \mid \nabla c_i(x)^\top (E\tilde{w} + E_\perp v_\perp) \geq 0,\ i \in \mathcal{A}(x) \},$$

which defines linear constraints on $\tilde{w}$, yielding a convex quadratic program. Therefore, we have the Karush-Kuhn-Tucker (KKT) conditions:

$$\begin{aligned}
\tilde{w} - v_\mathcal{E} - E^\top \sum_{j \in \mathcal{A}(x)} \lambda_j \nabla c_j(x) = 0, \\
0 \leq \lambda_j \perp \nabla c_j(x)^\top (E\tilde{w} + E_\perp v_\perp) \geq 0,
\end{aligned}$$

for $j \in \mathcal{A}(x)$ which are both necessary and sufficient. Solving the first equation for $\tilde{w}$ and substituting it into the expression for $\dot{x}$ yields:

$$\dot{x} = f(x,u) + EE^\top \sum_{j \in \mathcal{A}(x)} \lambda_j \nabla c_j(x),$$

which in combination with the complementarity conditions (also substituting the expression for $\tilde{w}$) can be rewritten as:

$$\begin{aligned}
&\dot{x} = f(x,u) + EE^\top \nabla c(x)\lambda \\
&0 \leq \lambda_j \perp \nabla c_j(x)^\top \dot{x} \geq 0, \\
&\lambda_l = 0, \quad l \in \{1, \ldots, n_c\} \setminus \mathcal{A}(x),
\end{aligned}$$

for $j \in \mathcal{A}(x)$. Note that $\nabla c_j(x)^\top \dot{x} = \dot{c}(x)$. The final condition on λ is equivalent to $0 \leq \lambda \perp c(x) \geq 0$. This is therefore equivalent to the DCS in eq. (6). □

With this we have a computationally useful equivalent DCS to which we can apply the previously developed FESD method.

IV. Time Varying FOSwP as ePDS

In this section we discuss the reformulation of FOSwP as an ePDS in order to apply the results described in the previous section. First, a general reformulation of the FOSwP to an ePDS is presented and the equivalence is verified. This section concludes with an example reformulation of a simple FOSwP to the ePDS.

In the following, the set $\mathcal{C}$ is considered to be time-dependent and finitely defined by $c : \mathbb{R}^{n_x} \times \mathbb{R} \to \mathbb{R}^{n_c}$.

A. FOSwP reformulation

We now return to the First-Order Sweeping Process (FOSwP): $\dot{x} \in f(x,u) - \mathcal{N}_{\mathcal{C}(t)}(x)$ and the definition of the finitely-defined time dependent set $\mathcal{C}(t) = \{ x \in \mathbb{R}^n \mid c(x,t) \geq 0 \}$ with $c : \mathbb{R}^{n_x} \times \mathbb{R} \to \mathbb{R}^{n_c}$. The mapping from $t \in [0,T]$ to $\mathcal{C}(t)$ is assumed to be forward Lipschitz [20, Definition 2] as $c(x,t)$ is assumed to be at least once continuously differentiable in x and Lipschitz continuous in t. The normal cone to this set at time t is the finitely defined cone $\mathcal{N}_{\mathcal{C}(t)}(x) = \{ d \in \mathbb{R}^{n_x} \mid d = \sum_{i \in \mathcal{A}(x)} \lambda_i \nabla c_i(x,t),\ \lambda_i \geq 0 \}$ [21]. We reformulate the FOSwP into an ePDS by introducing an auxiliary clock state τ, the dynamics of which we do not project. This allows us to accurately discretize the system using FESD.

The equivalence of an ePDS and the original sweeping process is formalized in theorem 5. To aid in the proving of this theorem we require the following lemma:

Lemma 4. *For $\mathcal{C}(t)$ uniformly prox-regular and Locally-Lipschitz [22] and $f(x,u)$ at least once differentiable there exists a unique absolutely continuous solution to the sweeping process eq.* (3) *for initial condition $x(0) \in \mathcal{C}(0)$ which corresponds to the solution to the DCS:*

$$\dot{x} = f(x,u) + \nabla_x c(x,t)\lambda \tag{7a}$$

$$0 \leq \lambda \perp c(x,t) \geq 0 \tag{7b}$$

The proof of this lemma follows from the transformation in [22, Section 4.2].

Theorem 5. *The solutions of eq.* (3) *with $\mathcal{C}(t)$ uniformly prox-regular and Locally-Lipschitz [22], and $f(x,u)$ continuously differentiable, correspond to the first component of the solutions to ePDS:*

$$\dot{y} = \mathrm{P}_{\mathcal{T}_\mathcal{C}(y),\mathcal{E}}\left((f(x,u),1)\right), \tag{8a}$$

$$y(0) = (x(0),0) \in \mathcal{C}, \tag{8b}$$

with state $y = (x,\tau) \in \mathbb{R}^{n_x+1}$, functions $\hat{c}(y) = c(x,\tau)$, $\hat{c} : R^{n_x+1} \to \mathbb{R}^{n_c}$, $\mathcal{C} = \{ y \in \mathbb{R}^{n_x+1} \mid \hat{c}(y) \geq 0 \}$, and subspace $\mathcal{E} = \{ s \in \mathbb{R}^n \mid EE^\top x = s,\ x \in \mathbb{R}^n \}$, with $E = (e_1, \ldots, e_n) \in \mathbb{R}^{(n_x+1) \times n_x}$ where e_i is the ith basis vector of $\mathbb{R}^{n_x+1}$.

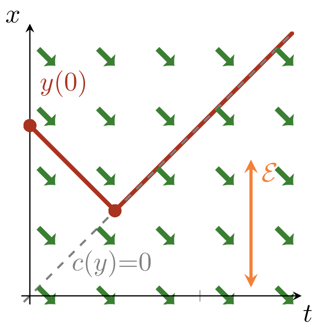


Fig. 4: example 1 ePDS trajectory.

Proof. As a consequence of lemma 4 we have an equivalent DCS for eq. (3). By theorem 3, for $y(0) = (x(0), 0) \in \mathcal{C}$ the solution to eq. (8) is the solution to the DCS:

$$\dot{y} = (f(x,u), 1) + EE^\top \nabla_y \hat{c}(y)\lambda'$$
$$0 \leq \lambda' \perp \hat{c}(y) \geq 0$$

From the structure of $\mathcal{E}$, $\dot{\tau} = 1$ and therefore $\tau(t) = t$. The fact that $EE^\top \nabla_y \hat{c}(y) = \nabla_x c(x,t)$ given $\tau(t) = t$ for a given x comes directly from the fact that the last row and column in $EE^\top$ is zero. This means that the solution to eq. (8) can, by substituting the above two equivalences, be decomposed:

$$\dot{x} = f(x,t) - \nabla_x c(x,t)\lambda',$$
$$\dot{\tau} = 1$$
$$0 \leq \lambda' \perp c(x,t) \geq 0$$

which is eq. (7) with an autonomous ODE for the numerical time τ. □

B. FOSwP reformulation example

We now provide a representative tutorial example for this reformulation and show that the solution for the ePDS reformulation is indeed the solution to the FOSwP.

Example 1. *Consider a sweeping process as in eq. (3) with $x \in \mathbb{R}$, $\mathcal{C}(t) = \{x \in \mathbb{R} \mid c(x,t) = x - t \geq 0\}$, $x(0) = 2 \in \mathcal{C}(0)$, and $f(x,t) = -1$. The solution to this sweeping process is the piecewise linear function:*

$$x(t) = \begin{cases} 2 - t, & 0 \leq t < 1, \\ t, & 1 \leq t, \end{cases}$$

as the state travels in the negative direction at a constant rate until it meets the boundary of $\mathcal{C}$ at $t = 1$ and then is swept by this boundary in the positive direction at a constant rate. We now reformulate this into an ePDS with $y = (x, \tau) \in \mathbb{R}^2$, $c(y) = x - \tau$, $f(y,t) = (-1, 1)$, and $E = (1, 0)$. This yields the projection matrix:

$$EE^\top = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

The normal cone of $\mathcal{C}$ in this case is defined as:

$$\mathcal{N}_\mathcal{C}(y) = \begin{cases} \emptyset, & y \notin \mathcal{C}, \\ \{0\}, & c(y) > 0, \\ \{\, d \in \mathbb{R}^2 \mid d = (1,-1)\lambda,\ \lambda \geq 0 \,\}, & c(y) = 0, \end{cases}$$

and the tangent cone:

$$\mathcal{T}_\mathcal{C}(y) = \begin{cases} \emptyset, & y \notin \mathcal{C}, \\ \mathbb{R}^2, & c(y) > 0, \\ \{\, (a,b) \in \mathbb{R}^2 \mid a \geq b \,\}, & c(y) = 0. \end{cases}$$

The solution to this system is again piecewise linear with a switch at $t = \tau(t) = 1$. For the first section of the trajectory the tangent cone is $\mathbb{R}^2$ therefore $y = (2 - t, t)$ while $c(y) = 2 - t - t > 0$, i.e. $t < 1$. At $t = 1$ the tangent cone becomes $\{\, (a,b) \in \mathbb{R}^2 \mid a \geq b \,\}$ which is a half-plane. Note that if we use the Euclidean projection operator, the resulting derivative $\dot{y}$ would be $(0,0)$. However, as we only allow projection along the x direction the closest point in $\mathcal{T}_\mathcal{C}(x)$ along $\mathcal{E}$ is the vector $(1,1)$. This vector is exactly tangential to the line $x - \tau = 0$ and for all remaining time $t \geq 1$: $c(y) = 0$ and $\dot{y} = (1,1)$. From this we get that the evolution of the first element of y matches exactly the evolution of the state of the FOSwP. A graphical representation the trajectories can be seen in fig. 4.

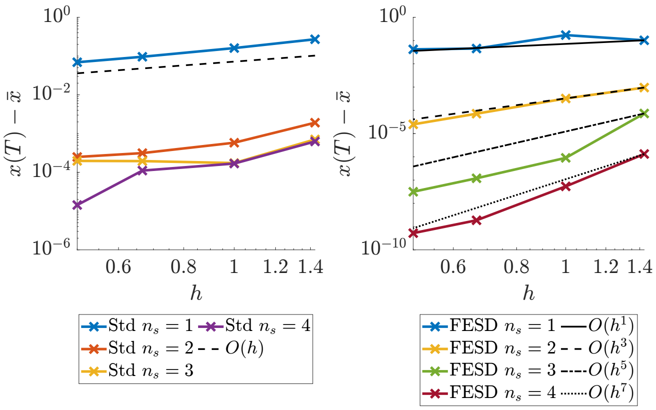
V. Finite Elements with Switch Detection

Direct optimal control functions on a "first discretize, then optimize" paradigm. As such, to apply this technique we must discretize the nonsmooth DCS in eq. (6). However, applying standard Runge-Kutta (RK) schemes to the system yields low accuracy, related to the order of the discontinuity [23], [24]. In this case, the system is either first order or second order discontinuous [25], i.e. either the first or second derivative of the state x is discontinuous, which corresponds to entering the boundary of $\mathcal{C}$ and leaving it, respectively. This yields, at best, $O(h^2)$ error where h is the step size. To improve on this accuracy we extend the recently introduced FESD for PDS method to this class of extended PDS [15].

We consider the trajectory entering or exiting the boundary of $\mathcal{C}$ as a "switch". This method can be directly applied to the DCS in eq. (6) as it demonstrates the same switching behaviors described in [15, Proposition 2]. Therefore, the FESD discretized system with N_{fe} integration steps, called finite elements, using an RK discretization scheme with n_s stages with Butcher tableau [26], $a_{i,j}$, b_j, c_i for $i, j = 1, \ldots, n_\text{s}$ is written as:

$$x_{n,i} = x_{n,0} + h_n \sum_{j=1}^{n_\text{s}} a_{i,j} v_{i,j}, \tag{9a}$$
$$v_{i,j} = f(x_{n,j}, u) + EE^\top \nabla c(x_{n,j})\lambda_{n,j} \tag{9b}$$
$$0 \leq c(x_{n,i}) \perp \lambda_{n,i} \geq 0, \tag{9c}$$
$$0 \leq c(x_{n,i}) \perp \lambda_{n,j} \geq 0, \tag{9d}$$
$$0 \leq c(x_{n-1,n_\text{s}}) \perp \lambda_{n,j} \geq 0, \tag{9e}$$
$$0 \leq c(x_{n,i}) \perp \lambda_{n-1,n_\text{s}} \geq 0, \tag{9f}$$

with $n = 1, \ldots, N_{\text{fe}}$, $i = 1, \ldots, n_\text{s}$, and $j = 1, \ldots, n_\text{s}$, and a fixed control u. Note that in this case h_n, the step size, is allowed to vary which allows the complementarity conditions, 9c through 9f, to be satisfied. The complementarity constraints here enforce that the active set $\mathcal{S}(x) = \{\, i \in \{1, \ldots, n_c\} \mid c_i(x) = 0 \,\}$ stays constant over each



(a) Fixed time stepping. (b) FESD.

Fig. 5: Plots of terminal error vs step size.

finite element. This is augmented with the step-equilibration constraints in [15, III.A] which enforce that $h_n \neq h_{n-1}$ only if there is a switch on the boundary between finite elements n and $n-1$. This discretization accurately identifies switches in the PDS active set and recovers the higher order accuracy of the RK discretization scheme. Using a higher accuracy discretization allows for a coarser control grid, while maintaining good constraint satisfaction and accuracy. This discretization is essentially the same as the one found in [15] with the $EE^\top \nabla c(x)\lambda$ term replacing $\nabla c(x)\lambda$, and more details can be found there.

VI. Numerical Experiments

In this section we first describe an optimal control problem, with which we demonstrate the numerical efficiency of the FESD discretization applied to a first-order sweeping process. We then provide an example optimal panning problem with dynamics defined by perturbed sweeping processes with time dependent sets. These examples are implemented in `nosnoc` [17] and solved with the Scholtes' relaxation [27] and homotopy loop using IPOPT [28] to a complementarity tolerance of 10^{-10} [29].

A. Wave-rider Problem

This example is a simple optimal control problem (OCP) over a perturbed sweeping process. The continuous-time OCP is formulated as:

$$
\begin{aligned}
\min_{x(\cdot).u(\cdot)} \quad & \int_0^T u(t)^2 \mathrm{d}t \\
\text{s.t.} \quad & x(t) \in (0, u(t)) - \mathcal{N}_{\mathcal{C}(t)}\left(x\right), \; t \in [0,T], \\
& x(0) = \hat{x}, \\
& x(T) = \bar{x}, \\
& -1 \le u(t) \le 1,
\end{aligned}
$$

with $x \in \mathbb{R}^2$, $u \in \mathbb{R}$, $\hat{x} = (0,2)$, $\bar{x} = (1,4)$, and time-varying set $\mathcal{C}(t) = \{\, z \in \mathbb{R}^2 \mid z_1 + \cos(z_2 - t) \ge 0 \,\}$. The system is under-actuated and the controller must use the moving boundary defined by $\mathcal{C}(t)$ to translate in the unactuated direction. We run an experiment with $N = [5, 10, 25, 50, 100]$ control stages using both the FESD discretization and the standard discretization ($n_s = [1,2,3,4]$ stage Radau-IIA integration steps) to show the efficiency of the FESD method applied to such problems. fig. 5 shows that even with few control stages the problem discretized with FESD yields a significantly lower terminal error ($\|x(T) - \bar{x}\|_2$) when accurately simulated using the calculated control. The control and state trajectory of this OCP can be seen in fig. 6.

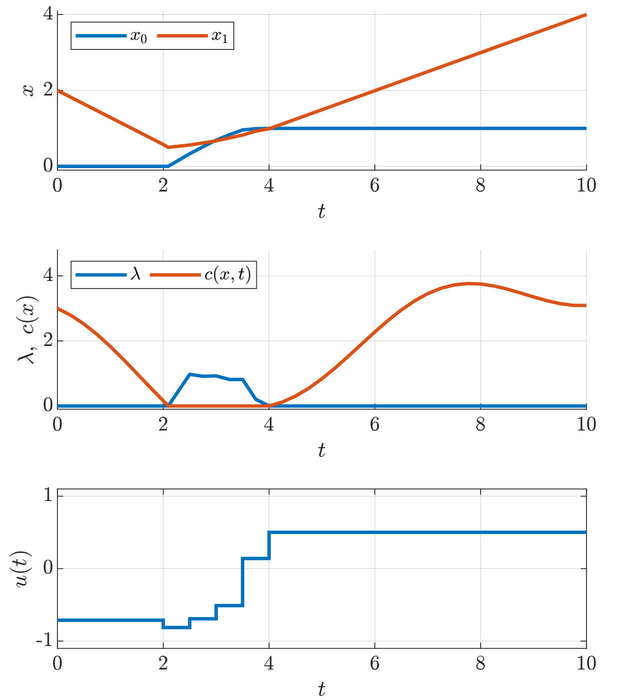


Fig. 6: Plot of wave-rider example trajectory.

B. Path Planning with Moving Obstacles

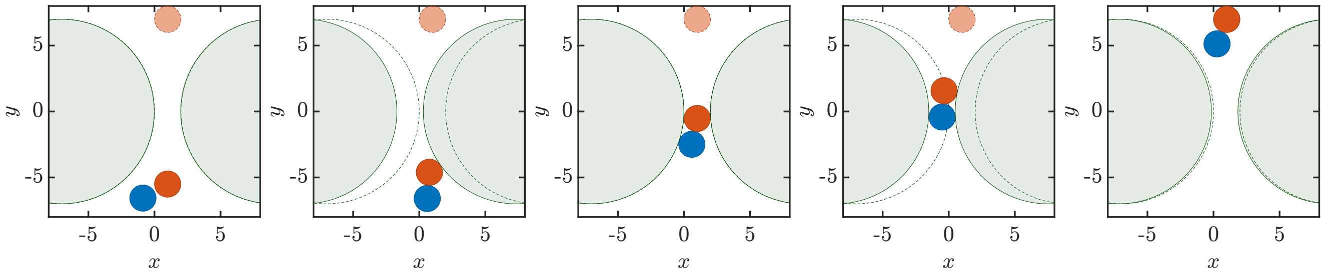
In this OCP, we demonstrate a manipulation example with a moving doorway made up of two discs whose centers follow a sinusoidal pattern. In this problem the goal is for an actuated pusher to push a disc through the moving doorway. The continuous time OCP is written as:

$$
\begin{aligned}
\min_{x(\cdot),u(\cdot)} \quad & \int_0^T u(t)^2 \mathrm{d}t + 1000\|(x_2, x_3) - (1,7)\|^2 \\
\text{s.t.} \quad & x(t) \in (0,0,u(t)) - \mathcal{N}_{\mathcal{C}}\left(x\right), \; t \in [0,T], \\
& x(0) = \hat{x}, \\
& -5 \le u_i(t) \le 5,
\end{aligned}
$$

with $x \in \mathbb{R}^4$, $u \in \mathbb{R}^2$, $\hat{x} = (-1,-6,1,-5)$, and time-varying set $\mathcal{C}(t) = \{\, z \in \mathbb{R}^4 \mid c(z,t) \ge 0 \,\}$ with:

$$
\begin{aligned}
c_1(x,t) &= \|(x_1,x_2) - (x_3,x_4)\|_2^2 - 2^2, \\
c_2(x,t) &= \|(x_1,x_2) - (\cos(t)+8, 0)\|_2^2 - 8^2, \\
c_3(x,t) &= \|(x_1,x_2) - (\cos(t)-8, 0)\|_2^2 - 8^2, \\
c_4(x,t) &= \|(x_3,x_4) - (\cos(t)+8, 0)\|_2^2 - 8^2, \\
c_5(x,t) &= \|(x_3,x_4) - (\cos(t)-8, 0)\|_2^2 - 8^2,
\end{aligned}
$$

which represent contacts between the unactuated slider, the actuated pusher, and the autonomous discs. These functions



(a) $t = 0$ (b) $t = 2.7001$ (c) $t = 6.3913$ (d) $t = 9.0000$ (e) $t = 12.0000$

Fig. 7: Several frames of the solution for the manipulation problem with autonomous discs.

form a moving aperture the size of the pusher and slider that follows a sinusoidal motion in the horizontal axis. Several frames of the solution to this problem are given in fig. 7. Full solutions and several further examples are available at https://youtu.be/hD1VMQGFEfw.

VII. Conclusions

In this paper, we show that under certain conditions the solutions to ePDS correspond to solutions of a dynamic complementarity system. We then show that Moreau's first-order sweeping processes with finitely defined moving sets can be re-formulated as an ePDS with a clock state whose dynamics are not projected. These reformulations allow us to use a lightly modified form of the finite elements with switch detection discretization to accurately discretize Moreau's first order sweeping processes. The approaches discussed are implemented in the open source software package `nosnoc`.

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